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# ON IRREGULAR BEHAVIOR OF NEURON SPIKE TRAINS

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## Abstract

The computational analysis of neuron spike trains shows that the changes in monotony of interspike interval values can be described by a special type of real numbers. As a result of such an arithmetical approach, we establish the presence of chaos in neuron spike trains and arrive at the conclusion that in stationary conditions, brain activity is found asymptotically close to a multidimensional Cantor space with zero Lebesgue measure, which can be understood as the brain activity attractor. The self-affinity, power law dependence, and computational complexity of neuron spike trains are also briefly examined and discussed.

## 1. INTRODUCTION

There are, in principle, two different approaches to investigating neural activity.[1] The traditional approach is based on the supposition of a random probabilistic character of neuron firings and makes use of the theory of stochastic processes. Indeed, the histograms of interspike intervals usually demonstrate an exponential decrease, that is consistent with the assumption of Poisson distributions. However, modern investigations of neuron spike trains prefer the deterministic point of view and explain their complex nature by the presence of deterministic chaos as suggested by nonlinear models of neuron activity. Thus, the exponential decay of histograms may still exhibit deterministic

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dynamics (see details in Refs. 1 and 2). The presence of correlations in neuron spike trains is more essential, as it contradicts one of the determining axioms of Poisson processes (see e.g. Ref. 3).

This paper is based on a method from Shahverdian's work.[4] All our conclusions are consequences of this method and the computational analysis of experimental data from single unit electrophysiologic recordings. These experiments were performed in the anesthetized monkey. All the neurons studied were located in the lateral thalamus and were characterized as wide dynamics range type neurons based on their responses to somatic adequate stimuli (for more details see Ref. 5). The neuron spike trains corresponding to two kinds of sustained stimuli applied to the same point of skin — pressure and pinch — as well as the spontaneous activity records, were collected and studied. We restrict ourselves only to a computational analysis of neuron spike trains (with average length $\approx 400$ spikes) and establish the presence of their properties, intrinsic to nonlinear dynamics.

The numerical study of interspike intervals' monotone increases and decreases, led us to the consideration of the existence of fast oscillations of spike train terms as our major standpoint. Since further analysis shows that such oscillations are also observed for higher order finite differences of neuron spike trains, it makes the application of Shahverdian's approach to these problems possible. In Sec. 2 below, the main statements of the finite-difference method, which in fact is a "thin" asymptotical modernization of Poincare classical method of dynamical systems research, are given. The large extent of universality of this approach for analysis of one-

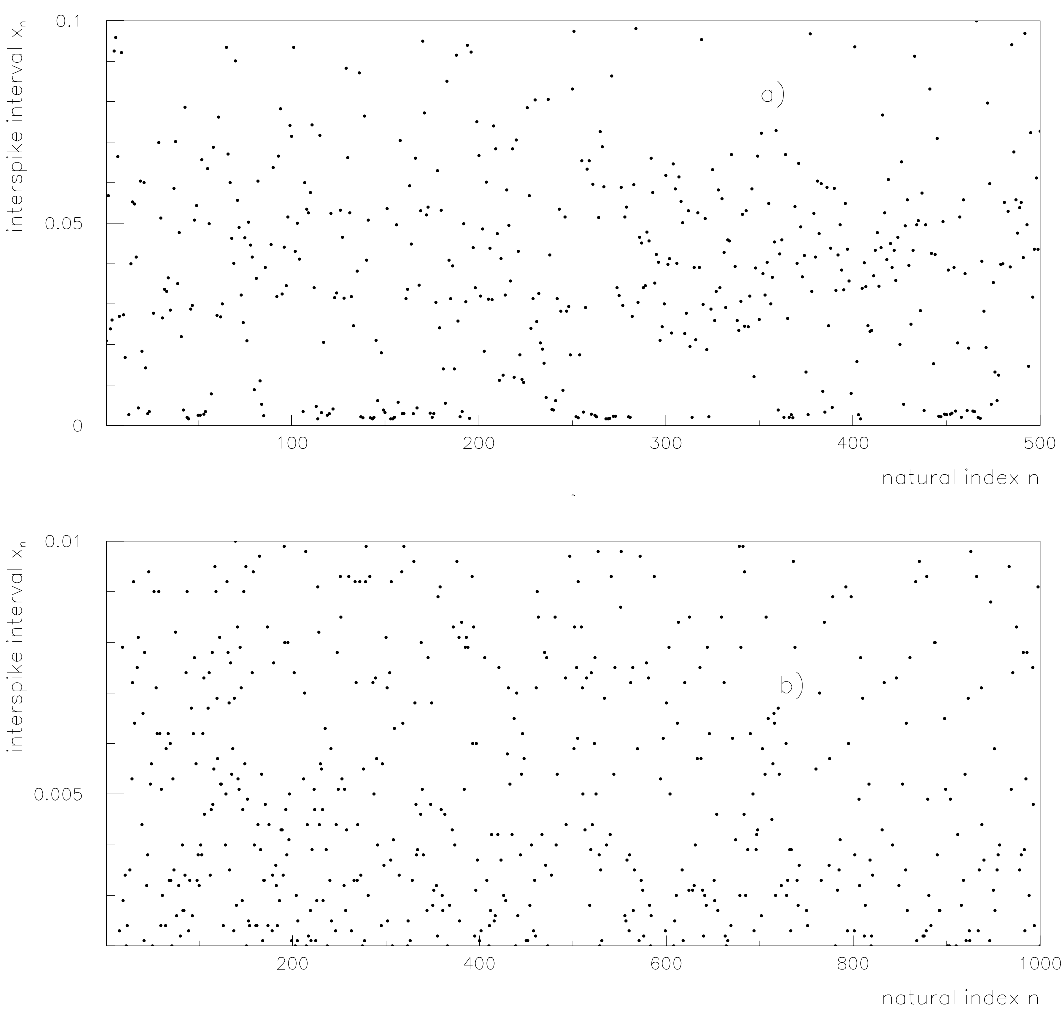


**Fig. 1** The random and unpredictable character of neuron firings. **(a)** The records of spontaneous neuron spike train. **(b)** The records of stimulated neuron. In both cases, almost the same chaotic picture is presented.

dimensional nonlinear systems is almost evident and in Sec. 3 we use it to analyze neuron spike intervals. In this way, we establish that in stationary conditions, brain activity is found asymptotically close to a multidimensional Cantor set with zero Lebesgue measure.

In Sec. 4, the self-affinity of interspikes records, that is the characteristic property of records of such processes as Brownian motion, is briefly examined. We also establish the partial power law dependence of some function easily determined via distribution of interspike intervals, measuring its fractality on a wide range of argument values. The applicability of some introduced quantities to differentiation of various modes of a given neuron, the possibility of real-time computer generation of neuron spike trains as well as their computational complexity, are discussed. Shahverdian's method not only allows the detection of chaos in the behavior of spontaneous activity, it seems far more promising as well. For example, in spite of almost the same chaotic picture on graphs of spontaneous and stimulated interspike interval records (see Fig. 1), the periodicity in changes of monotony of (all of sufficient higher order) absolute finite differences of stimulated neuron's interspike intervals has been observed very distinctly. This allows us to introduce in Sec. 4, the rigorous definition of (spontaneous and stimulated) neuron spike trains. Regretfully, the deficiency in information on spike trains does not allow the application of the full power of this method.

## 2. A METHOD OF ANALYSIS OF ONE-DIMENSIONAL SYSTEMS

In this section, we introduce some notions and main statements of Shahverdian's paper, which we use in the Secs. 3 and 4. We consider the numbers $0 < x < 1$ represented in the form of binary expansion,

$$x = 0.\delta_1\delta_2\delta_3\ldots \left(= \sum_{n=1}^{\infty} 2^{-n}\delta_n, \quad \delta_n = 0, 1\right) \quad (1)$$

as well as the subsets $\mathcal{B}_k$ of numerical interval $(0, 1)$, are defined as follows: for given $k \geq 2$, the $\mathcal{B}_k$ is the set of all those real numbers $x \in (0, 1)$ for each of which

$$n_{i+1} - n_i \leq k \qquad (i = 1, 2, 3, \ldots) \quad (2)$$

where $n_i$ denote all the consecutive positions where the changes of binary symbol from Eq. (1) occur, $\delta_{n_{i+1}} = 1 - \delta_{n_i}$. Let also $\mathcal{B}$ be the union of all $\mathcal{B}_k$.

Let us have an infinite numerical sequence $\bar{X} = (x_i)_{i=1}^{\infty}$ of numbers $x_i$ from unit interval $(0, 1)$, generated by some one-dimensional system. For example, it can be the sequence of consecutive iterates of some map defined in $(0, 1)$, but in general, we impose no restrictions on mechanism generating $\bar{X}$. Particularly, the generating system may possess different inner states, changing with time. We formulate below, two conditions under which the sequence $\bar{X}$ should be regarded as chaotic. For the finite sequence $\bar{X}_k = (x_i)_{i=1}^{k}, \ \ k \geq 2$ and number $1 \leq s \leq k - 1$, we let

$$\Delta_i^{(0)} = x_i \, ,$$
$$\Delta_i^{(s)} = |\Delta_{i+1}^{(s-1)} - \Delta_i^{(s-1)}| \quad (i = 1, 2, \ldots, k - s) \, .$$

It is not difficult to obtain

$$\Delta_i^{(s-1)} = \mu_{k,s-1} + \sum_{p=1}^{i-1} (-1)^{\delta_p^{(s)}} \Delta_p^{(s)} - \min_{0 \leq i \leq k-s} \left( \sum_{p=1}^{i} (-1)^{\delta_p^{(s)}} \Delta_p^{(s)} \right) \quad (3)$$

where is supposed that $\sum_1^0 = 0$, and

$$\mu_{k,s} = \min\{\Delta_i^{(s)} : 1 \leq i \leq k - s\}$$

and

$$\delta_p^{(s)} = \begin{cases} 0 & \Delta_{p+1}^{(s)} \geq \Delta_p^{(s)} \\ 1 & \Delta_{p+1}^{(s)} < \Delta_p^{(s)} \, . \end{cases} \quad (4)$$

We can consider that the finite sequence $\bar{X}_k$ is given in the next special representation

$$\bar{\zeta}_k = (r_1^{(k)}, r_2^{(k)}, \ldots, r_m^{(k)}; \mu_{k,1}, \ \mu_{k,2}, \ \ldots, \mu_{k,m}; \rho_1, \ \rho_2, \ \ldots, \rho_{k-m}) \quad (5)$$

where

$$r_s^{(k)} = 0.\delta_1^{(s)} \delta_2^{(s)} \cdots \delta_{k-s}^{(s)} \qquad (1 \leq s \leq m) \quad (6)$$

are the (binary -) rational numbers and

$$\mu_{k,1}, \ \mu_{k,2}, \ \ldots, \mu_{k,m}$$

and

$$\rho_1, \ \rho_2, \ \ldots, \rho_{k-m} \quad (\text{where} \quad \rho_i = \Delta_i^{(m)}) \quad (7)$$

are some numbers from interval $[0,1]$, $m = m_k$ are some numbers which tend to $\infty$ as $k \to \infty$. Indeed, the recurrent procedure (3) allows the complete restoration of $\bar{X}_k$ through $\bar{\zeta}_k$. We say, that numerical sequence $(\nu_s)_{s=1}^{\infty}$, is the conjugate orbit, associated with $\bar{X} = (x_i)_{i=1}^{\infty}$, if for each $s \geq 1$, its terms $\nu_s$ are defined as follows

$$\nu_s(\bar{X}) = \nu_s = \lim_{k\to\infty} r_s^{(k)} \ (= 0.\delta_1^{(s)}\delta_2^{(s)}\delta_3^{(s)}\ldots) . \quad (8)$$

Let us assume, that the orbit $\bar{X}$ is such, that each of the sequences

$$\delta_1^{(s)}, \delta_2^{(s)}, \ldots, \delta_n^{(s)}, \ldots \quad (s = 0, 1, \ldots) \quad (9)$$

[see Eq. (6)] has bounded lengths of series with the same binary symbol, or in other words, for increasing sequence of indices $n_i^{(s)}$, designating all of those positions in natural series where the changes of binary symbol occur, $\delta^{(s)}_{n_{i+1}^{(s)}} = 1 - \delta^{(s)}_{n_i^{(s)}}$, we have

$$n_{i+1}^{(s)} - n_i^{(s)} \leq \text{const.} < \infty \qquad (i = 1, 2, \ldots) . \quad (10)$$

The next restriction on the sequence $\bar{X}$ is that the quantities

$$\mu(\bar{X}_k) = \sum_{i=1}^{m_k} \mu_{k,i}^2 + \sum_{i=1}^{k-m_k} \rho_i^2 \quad (11)$$

converge to zero,

$$\mu(\bar{X}_k) = o(1) \qquad (k \to \infty) . \quad (12)$$

Then, if considering that $\mathcal{B}^1 \subset \mathcal{B}^2 \subset \cdots \subset \mathcal{B}^\infty$, the transformed sequence $\bar{\zeta}_k$ from Eq. (5) tends to the space $\mathcal{B}^\infty$,

$$||\bar{\zeta}_k - \mathcal{B}^\infty|| = o\ (1)$$
$$\left(k \to \infty, \quad ||x|| = \left(\sum_{i=1}^{k} x_i^2\right)^{1/2}\right) . \quad (13)$$

In such a way, the conditions (10) and (12) imply that $\mathcal{B}^\infty$ is the attractor of the orbits $\bar{\zeta}_k$. Furthermore, let $\mathcal{A}$ be the union of the set $\{\nu_s : s \geq 1\}$ with the collection of all its cluster points. Then the orbit $\bar{X}$ should be considered as a chaotic one, whenever $\mathcal{A}$ is a Cantor set — this requirement is in complete co-ordination with that of the $n$ dimensional ($n \geq 2$) systems. If $\mathcal{A}$ appears to be the same set for almost each $\bar{X}$, then we can conclude that $\mathcal{A}$ is the attractor of the system.[6] If for all $s \geq 1$, the constants from Eq. (10) are upper bounded by the same number $K \geq 2$, then in all the relations above, the set $\mathcal{B}$ can be replaced by set $\mathcal{B}_K$. This remark implies the upper estimate of Hausdorff dimension,[7] $\dim(\mathcal{A})$ of attractor $\mathcal{A}$. Since $\nu_s \in \mathcal{B}_K$ and $\mathcal{B}_K$ is a compact set, we deduce

$$\mathcal{A} \subset \mathcal{B}_K \quad \text{and hence} \quad \dim(\mathcal{A}) \leq \dim(\mathcal{B}_K) . \quad (14)$$

The Cantor sets $\mathcal{B}_k$ permit the exact estimates of their Hausdorff dimension (see Ref. 4): $\dim(\mathcal{B}) = 1$ and

$$\dim(\mathcal{B}_k) = \log_2 z_k^{-1}(1)$$

where

$$z_k(s) = \sum_{n=1}^{k} \frac{1}{s^n} \quad (s > 0) \quad (15)$$

note also, that all the sets $\mathcal{B}$ and $\mathcal{B}_k$ possess zero Lebesgue measure (see Ref. 4).

Since analytical criteria for the checking of the justice of the determining relations (10) and (12) are not provided, this method of establishing the chaos, generally speaking, preassumes the computational study of the given system.

## 3. CHAOS IN NEURON SPIKE TRAINS

The typical record of interspike interval sequence of the neuron's firings (Fig. 1) represents some irregular set of points in a two-dimensional plane. To each value of natural argument, $1 \leq i \leq k$ ($k$ is the total number of records in the spike train) corresponds to some number $x_i > 0$ that is the length of interspike interval between neuron's firing in $i$th and $(i+1)$th time. The numerical sequence $\bar{X}_k = (x_i)_{i=1}^{k}$ is the neuron spike train. It is hardly possible to detect any regularity in behavior of sequence $x_i$. The only thing one can see immediately on this graph are the fast oscillations of this function. This feature of interspike intervals is our starting point and is further studied below.

In order to apply the analysis from Sec. 2 to research on neuron and brain activity, let us have a neuron spike train $\bar{X}_k = (x_i)_{i=1}^{k}$ (where $k$ is the total number of records) and let

$$\bar{X}_k^s = (\Delta_i^s)_{i=1}^{k-s} \quad (16)$$

be the sequence of (absolute) finite differences of order $s \geq 0$ taken from spike train $\bar{X}_k$ $(= \bar{X}_k^0)$; for

$n = 1, 2, \ldots, k-s$, we also let

$$\delta_n(\bar{X}_k^s) = \begin{cases} 0 & \Delta_n^s \leq \Delta_{n+1}^s \\ 1 & \Delta_n^s > \Delta_{n+1}^s \,. \end{cases} \tag{17}$$

We have examined 30 spike trains of 10 different neurons each having 3 different modes (according to applied stimulation — pressure, pinch, and spontaneous activity). The result shows that for all 30 spike trains $\bar{X}_k$, the relation (10) is satisfied and, consequently according to Sec. 2 we can put in correspondence to each spike train $\bar{X}_k = (x_i)_{i=1}^k$, some numerical sequence $(\nu_s)_{s=1}^{m_k}$ (following Sec. 2 we call it the conjugate spike train). More accurately the computer-assisted analysis shows, that it is probably possible certain growth of the lengths $n_{i+1}^{(s)} - n_i^{(s)}$ from Eq. (10) goes to $\infty$ as $i \to \infty$. However, it is too weak and insignificant to be considered as a good approximate for the complete satisfaction of condition (10) by actual neuron spike trains. Furthermore, computations show that these lengths are upper bounded by some small constant $K$ (e.g. $K = 10$), which is common for all $s \geq 1$ and all 30 spike trains.

In addition, we have examined for our neural case, the justice of the condition (12) from Sec. 2. The results show that this condition always holds as well — the quantities $\mu(\bar{X}_k)$ defined by Eq. (12) converge to zero with exponential rate. As they become negligibly small, rapidly it is better to regard them as equal to zero than keep their exact numerical values for further operating. If we adopt the limitation of Eq. (12) on $\mu-$ and $\rho$-coordinates in special representation from Eq. (5) as the second determining property of actual neuron spike trains (which is confirmed well by computational analysis of experimental data), then the statements of Sec. 2 imply

$$||\bar{\zeta}_k - \mathcal{B}_K^\infty|| = o(1) \quad (k \to \infty)\,. \tag{18}$$

This relation means that for all the neurons where the quantities $\bar{\zeta}_k$ through which the complete restoration of the initial spike train is possible (see Sec. 2), converge to the same thin space $\mathcal{B}_K^\infty$. In other words, the set $\mathcal{B}_K^\infty$ is the attractor of special spike trains $(\bar{\zeta}_k)_{k=1}^\infty$.[4–8]

It is important to add that the conjugate train $\bar{\nu}_{m_k} = (\nu_i)_{i=1}^{m_k} \in \mathcal{B}_K^{m_k}$ for which the distance between $\bar{\zeta}_k$ and $\mathcal{B}_K^{m_k}$ in relation (18) is reached, together with their cluster points consist of some thin set. Indeed, computational study of all 30 spike trains shows that the next statement is true: for each conjugate train $(\nu_i)_{i=1}^\infty$, the set $B \subset \mathcal{B}_K \subset [0,1]$ of its cluster points is either a discrete or a Cantor set. This also follows from results of Sec. 2, if we take into consideration, the relation (14), which together with (15), gives us the estimate of Hausdorff dimension of $B$:

$$\dim(B) \leq \dim(\mathcal{B}_{10}) = 0.99933\ldots\,. \tag{19}$$

The graphs in Fig. 2 illustrate this statement — thin nature of sets $B$ has been observed for all 30 spike trains. Figure 2(a) demonstrates the convergence of conjugate train to a "wide" subset (a Cantor set) of interval $(0,1)$. In Figs. 2(b) and (c), the convergence to a "small" subset (a finite set and a single point respectively) of $(0,1)$ is shown. We note that Fig. 2(c) presents the graph of conjugate spike train of a strongly stimulated ("pinch") neuron, while for a case of spontaneous activity, the set $B$ always has "wide" extended Cantorian structure. The results of Shahverdian's paper demonstrate very clearly, that conjugate orbits of the logistic map $x \to rx(1-x)$, as introduced above, inherit the chaotic properties of the initial system. Hence, based on previous results, it can be said analogously that neuron spontaneous activity is always found in a chaotic state. Another consequence of the applied method is the following important observation: for each stimulated neuron, the binary sequences from Eq. (9) corresponding to all (having sufficient higher order $i$) finite differences become periodic. It means therefore that the presence of stimulation on neuron can be stated through consideration of higher order differences of spike train. Thus, the periodicity of sequences from Eq. (9) being a criterion for determination of the presence of a (periodic) stimulation. This statement (see also definitions from Sec. 4) probably has very important consequences and needs further detailed study.

Let us now apply this arithmetical approach to research of neurons and brain activity in the case when they function under certain stationary conditions, held for a sufficiently long time period $T$. More exactly, we consider that during this period, each neuron is either in spontanous mode or is subjected to the same stimulation with constant parameters. This requirement simply means that all neurons are found in conditions close to those under which our experimental data were obtained. It was mentioned above, that in limiting case $k \to \infty$, we can assign to each spike train

$$\bar{X}_k = (x_1, x_2, \ldots, x_k) \quad x_i \in (0,1) \quad (1 \leq i \leq k) \tag{20}$$

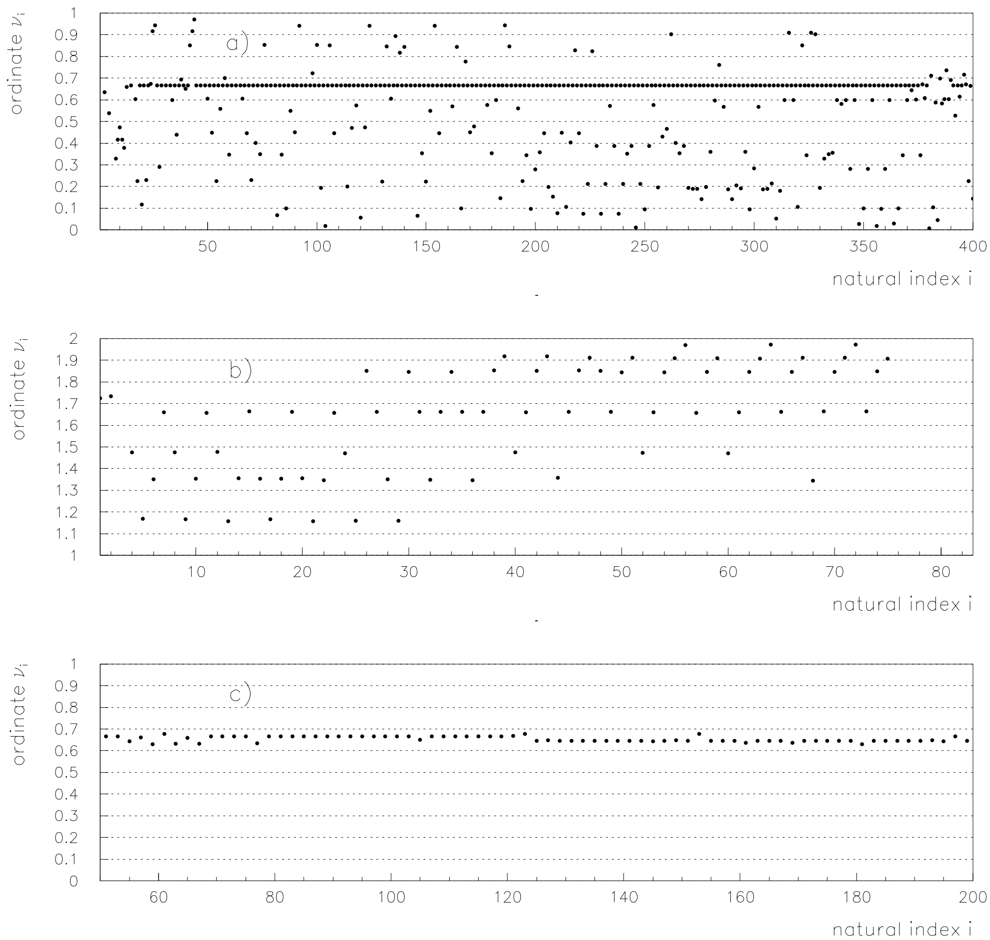


**Fig. 2** These graphs demonstrate the thin nature of cluster sets of conjugate spike trains. **(a)** The general case when cluster set $B$ of conjugate train is a "wide" subset (a Cantor set) of numerical interval $(0,1)$. **(b)** Attractor $B$ is a "small" subset (a finite set) of $(0,1)$. **(c)** The case (strongly stimulated neuron) when this cluster set consists of a single point.

some vector $\bar{\nu}_m$ $(= \bar{\nu}_m(\bar{X}_k))$ of the form

$$\bar{\nu}_m = (\nu_1, \nu_2, \ldots, \nu_m) \quad \nu_i \in \mathcal{B}_K \quad (1 \le i \le m) \tag{21}$$

of which all of the coordinates belong to $\mathcal{B}_K$ and the initial train $\bar{X}_k$ can be completely restored through terms of some vector $\bar{\zeta}_k$ such that the distance $||\bar{\zeta}_k - \bar{\nu}_m||$ tends to zero as $k \to \infty$; the length $m$ of $\bar{\nu}_m$ depends on $k$, $m = m_k$ and $m_k \to \infty$ as $k \to \infty$.

We note the difference between representations given by Eq. (20) and Eq. (21). Generally speaking, the terms $x_i$ of $\bar{X}_k$ can be arbitrary real numbers from interval $(0,1)$, i.e. $\bar{X}_k$ being a point of unit cube

$$D^k = [0,1] \times \cdots \times [0,1] \quad (= [0,1]^k)$$

whereas the terms $\nu_i$ of vector $\bar{\nu}_m$ possess a special property [bounded differences from Eq. (10) for binary expansion in Eq. (8)], by virtue of which $\bar{\nu}_m$ appears as a point of special space

$$\mathcal{B}_K^m = \mathcal{B}_K \times \cdots \times \mathcal{B}_K$$

here the dimension $m = m_k$ of Euclidian space $R^m$ into which $\mathcal{B}^m$ is embedded, depends on $k$. Thus, the analysis of experimental data allows to cut out from unit cube $D^k$, the arithmetical space $\mathcal{B}_K^m$ that already contains certain special (though incomplete) information on neuron spike trains. The quantities $\mu(\bar{X}_k)$ from Eq. (11) tend to zero so rapidly that for time period $T$ equal to several hours ($k \approx 10^5$), the relation (18) gives us certain non-trivial (though incomplete) information on location of neuron spike trains and consequently on location of brain states in the unit cube $D^\infty$. Hence for large $k$, the only useful (but incomplete) information [particularly on recovering the initial spike train through its special representation from Eq. (5)] is focused on $r$-coordinates of vector $\bar{\zeta}_k$ from Eq. (5), or similarly (by virtue of relation (18) and $k \to \infty$ provided) in space $\mathcal{B}_K^\infty$.

Due to certain resemblance in behavior of neuron spike trains and logistic map orbits (which has two attractors, for initial and conjugate orbits respectively[4]), both systems satisfy relations (10) and (12) and we conclude the existence of a brain activity attractor. Indeed, by virtue of Eq. (18) and the latter statement on sets $B$, one can say that each point of Cartesian product

$$\hat{\mathcal{B}}^* = B_1 \times \cdots \times B_\mu$$

is the cluster point of both $\mu$-dimensional special train $(\bar{\zeta}_m^{(1)}, \ldots, \bar{\zeta}_m^{(\mu)})_{m=1}^\infty$ and conjugate train $(\bar{\nu}_m^{(1)}, \ldots, \bar{\nu}_m^{(\mu)})_{m=1}^\infty$; here $\mu$ denotes the total number of neurons in a brain, $\bar{\zeta}_m^{(s)}$ and $\bar{\nu}_m^{(s)}$ are special and conjugate spike trains of the $s$th neuron respectively and, $B_s$ is the corresponding set defined above. This means that $\hat{\mathcal{B}}^*$ is an attractor (e.g. according to definition of attractor from Ref. 6) which should be considered as conjugate to some (brain activity) attractor $\hat{\mathcal{B}}$. Finally, it is noted that our analysis testifies that the volume of neuron activity attractors $B$ introduced above, apparently depends on intensity of applied stimulation and may vary (see also Fig. 2) from a single point on strong stimulation to the complete coincidence of $B$ with Cantor set $\mathcal{B}_K$ in the case of spontaneous activity.

The Hausdorff dimension of brain activity attractor $\hat{\mathcal{B}}$ can also measure the volume of the intellectual abilities of given biological species. With this approach, we have an interesting direction for brain research, where the methods and results of theory of fractals can be applied to the study of attractors $\hat{\mathcal{B}}$ and $\hat{\mathcal{B}}^*$.

## 4. SELF-AFFINITY, POWER LAW, AND COMPLEXITY OF NEURON SPIKE TRAINS

Unpredictability and irregularity of the neuron firings are two of the reasons for the application of deterministic chaos and fractals theory to its study. For example, the self-similarity of the neuron firing rate as well as the power law behavior of Fano-factor time curve[3] is discussed. The concept of self-affinity has been introduced by B. Mandelbrot (see e.g. Refs. 9 and 10) as a generalization of a more special case of self-similarity. This concept turns out to be a valuable tool particularly on investigation of records of a chaotic variable (see e.g. Refs. 10 and 11). We have applied it to the study of spike train records — namely, for a given spike train and natural number $L \geq 1$, we have studied the behavior of mean value

$$V(L,n) = \frac{1}{n}\sum_{i=1}^{n-L} |x_i - x_{i+L}|$$

(here we somewhat depart from traditional mean square deviation). The result of computations carried out on the 30 experimental spike trains, show that for each of these sequences, there is some constant $\alpha > 0$ such that the relation

$$V(L,n) \approx \frac{1}{L^\alpha}, \; n \to \infty \tag{22}$$

holds on a wide range of consecutive values $L$, separated from limiting ones 1 and $n$ (e.g. $40 \leq L \leq n - 40$). If one follows the generally accepted terminology,[12] Eq. (22) means that spike train records have a negative roughness exponent. The records of random walk and Brownian motion serve as examples of self-affine functions with positive roughness exponent.[11,12] Let us note the following consequence of relation (22) which is also applicable to general self-affine functions possessing the finite roughness exponent. The property of splittability of some chaotic deterministic systems was detected,[13] when the splitting of time series allows the revealing of more regular behavior of corresponding subsystems. In our case, we have a weaker statement — the complex stochastic process, governing the neuron firings, can be divided into an infinite number of non-trivial processes, which permit the convergence of variance increment mean value. Indeed, if for natural $L$ assigns $\rho_L(i) = x_{iL}$, $(i = 1, 2, \ldots)$, then from Eq. (22) we obtain

$$\frac{1}{n}\sum_{i=1}^{n} |\rho_L(i+1) - \rho_L(i)| \to \text{const.} \quad (n \to \infty)$$

which means that spike trains have a more regular behavior on each subsequence of natural series of the form $N_L = \{L, 2L, 3L, \ldots\}$. We note that contrary to results from Shahverdian,[13] $N_L$ and $N_{L'}$ $(L \neq L')$ may have non-empty intersections.

We have already mentioned some works researching the behavior of neuron's firing rate from the point of view of fractals theory. The next remark allows us to reveal the fractal nature of neuron spike trains more directly, which can be useful in further investigations of neuron activity. Let us suppose that at some moments $t$ and $t'$, two consecutive firings occur. During the period $(t, t')$, the cell membrane accumulates a potential electric energy, which in the end dissipates in the form of an electrical pulse, when it reaches threshold. If we assume that the accumulation of energy on time interval $(t, t')$ is (in average on time) proportional to the linear growth of time, we arrive at the conclusion that the length of interspike interval can also be treated as a value of energy, accumulated by neuron during the time period between two consecutive firings. It is known that such accumulation of potential mechanical energy is the main reason for seismic phenomena. Hence, it is natural to try to learn whether there exists an analogy to the seismologic Gutenberg-Richter law[14] for the neuronal case. This empirical law, describing the distribution of energy of seismic events with magnitude (energy) not exceeding some large value $s_{\max}$, has the power law form

$$\Phi(s) \approx \frac{\text{const.}}{s^{1+\gamma}}, \quad s \in (0, s_{\max}) \tag{23}$$

where $\Phi(s)$ is the total number of events with magnitude equal to $s$ and $\gamma$ is some numerical quantity bounded by narrow limits.

We note that the same assumption on linear growth with time of accumulated energy[15] applies to the neuron as well (see also Ref. 1). However, we do not use these arguments, but instead present them only as comments for the next problem: if for a given neuronal spike train where the $\Psi(s)$ designates the total number of interspike intervals with lengths $\geq s$, does this function show a power law behavior? Here, we have intentionally changed the sign "=" in definition of $\Phi(s)$ to "$\geq$", since the direct transference of seismic context does not lead to any reasonable results. It is not difficult to study this problem by computational methods. The result obtained can be formulated in the following form: there exists some constant $\eta > 0$ (e.g. $\eta = 1/3$), such that for each neuron there is some interval $I$ of values $s$ with the length $|I| \geq \eta k$ ($k$ is the total number of records), such that the function $\Psi(s)$ has the power law dependence on $I$,

$$\Psi(s) \approx \frac{1}{s^{1+\lambda}}, \quad s \in I\,. \tag{24}$$

The records of available data were collected with $10^{-4}$ precision and most of them were situated in interval $(0, 0.15)$, hence it is reasonable to assume that variable $s$ is represented in the form $s = i \times 10^{-4}$, $0 \leq i \leq 150$; here, the value $150 \times 10^{-4}$ may be interpreted as some analogy of limiting magnitude $s_{\max}$ from Eq. (23). Thus, the computations show that for each of 30 given spike trains, the relation (24) is true at least for 50 consecutive values of $s$. The constant $\lambda$ from Eq. (24) varies in dependence on each neuron as well as on the mode of its state, thus testifying the possibility of distinguishing neuron modes by numerical values of exponent $\lambda$. These values, averaged by 10 different neurons and corresponding to three considered modes, are

$$\lambda_{\text{spontan}} = 0.54 \pm 0.04\,,$$
$$\lambda_{\text{pressure}} = 0.66 \pm 0.04\,,$$
$$\lambda_{\text{pinch}} = 0.76 \pm 0.04\,.$$

Although the data are not large enough to make a final statistical statement, one can see quite a distinct difference between (averaged) numerical values of $\lambda$ in dependence on applied stimulation. Note that a similar analysis on dependence of Fano-factor power law exponent on stimulation for auditory-nerve spike train can be found in Teich and Lowen.[3] Another approach based on application of Kolmogorov–Martin-Löf–Chaitin algorithmic complexity[16] in Rapp et al.[17] is used. The partial power law obtained above, was also displayed in a computational study of some exact results on self-organized criticality;[18] on the possibility of application of self-organized criticality[19] models for investigation of neural and brain activity (see e.g. Ref. 3).

In order to examine the complexity and randomness of neuron spike trains, let us again have a neuron spike train $\bar{X}_k = (x_i)_{i=1}^k$ and let $\bar{X}_k^s$ be the sequence of (absolute) finite differences defined by Eq. (16). It is well known that the relation

$$\frac{1}{n}\sum_{i=1}^{n} \delta_i(x) \to 1/2 \quad (n \to \infty)\,, \tag{25}$$

where the functions $\delta_i(x)$ are defined in Eq. (1), holds for almost every (in sense of Lebesgue measure) real number $x$. Such $x$ is called Borel or weak normal[16,20] number. If one considers the quantities

$$\sigma(n, s, k) = \frac{1}{n}\sum_{i=1}^{n} \delta_i(\bar{X}_k^s) \quad (1 \leq n \leq k - s) \tag{26}$$

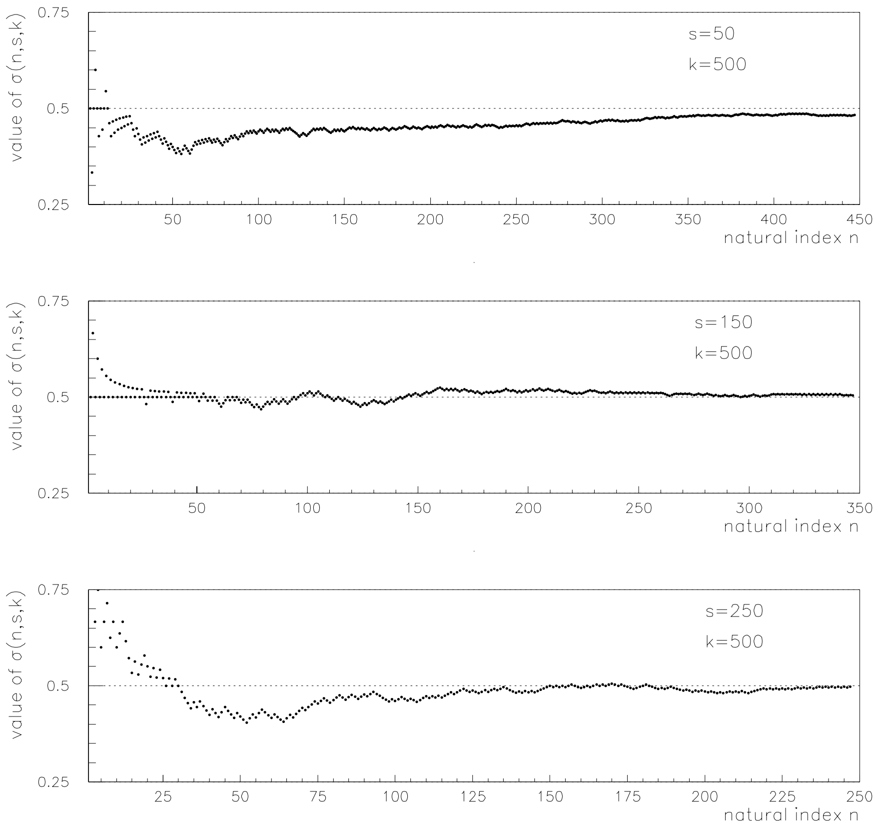


**Fig. 3** The presence of Borel property $\sigma(n,s,k) \to 1/2$ of conjugate spike train of spontaneous neuron. The length $n$ of initial spike train is equal to 500 (spikes), the graphs are constructed for 3 different values of $s$ denoting the order of finite differences taken from initial spike train.

[cf. Eq. (17) and Eq. (25)], after computations one can detect that

$$\sigma(n,s,k) \to 1/2 \tag{27}$$

for all

$$s = 0, 1, 2, \ldots, [k/2] \quad (\text{and } n, k \to \infty)\,.$$

Some results of these computations in Fig. 3 are given — the presence of Borel property expressed by Eq. (27) has been observed for all 10 neurons. Thus, letting $k$ tend to infinity and taking into attention the statements of Sec. 2, we have established the following two determining properties of (theoretical) infinite neuron spike trains:

(a) each of the numbers $\nu_i(\bar{X})$ $(i = 1, 2, 3, \ldots)$ defined by Eq. (8) belongs to $\mathcal{B}$ and represents the Borel number

(b) the quantities $\mu(\bar{X}_k)$ defined by Eq. (11) tend to zero quickly enough,

$$\mu(\bar{X}_k) = o(\mu_k) \quad (k \to \infty)$$

where convergent to zero, the given positive numbers $\mu_k$ do not depend on $\bar{X}$.

These two points can also be considered as a rigorous definition of theoretical neuron spike train: numerical sequence $\bar{X} = (x_i)_{i=1}^{\infty}$, $0 \le x_i \le 1$ is called the neuron spike train if the properties (a) and (b) are satisfied. Due to the above conclusions and the periodicity of sequence mentioned in Eq. (9), the next definitions of spontaneous and stimulated neuron spike trains can also be introduced: a single neuron is called spontaneous, if the sequence $\nu_i(\bar{X})$ $(i \ge 1)$ of numbers from point (a) contains infinite number of irrational terms; otherwise, it is called stimulated neuron. Thus, it is clear that the neuron spike trains possess very special number-theoretical properties. However, the construction of such numerical sequences is by no means evident.

The relation $\sigma \to 1/2$ from Eq. (27) reflects the complex (or random) nature of all of the numbers

$\nu_i(\bar{X})$ (or corresponding binary sequences $(\delta_n^{(i)})_{n=1}^{\infty}$ [see Eq. (9)]). Some authors (see Ref. 16) define the random binary sequence simply as some sequence $(\delta_i(x))_1^{\infty}$ from Eq. (1) where $x$ is a Borel number. However, it is normally accepted to prescribe to randomness some additional limitations. Thus, the property expressed by Eq. (25) is just the first part of Church's formulation of von Mises definition of randomness (see Ref. 21). In our case, checking out the second part is hardly possible. Nevertheless, one can make certain conditional statements on the possibility of generating artificial spontaneous neuron spike trains. Namely, let us show that through the consideration of conjugate trains, this problem can be reduced to some hypotheses from computations theory. Since these hypotheses remains unresolved, we can only make conditional conclusions. The notion of computational complexity of sequences has been introduced in Hartmanis.[22] The main result of this work, related to complexity of real numbers (given in the form of its binary expansions), establishes that each rational as well as some transcendental numbers are real-time computable (see details in Ref. 22). To our knowledge, the following Hartmanis-Stearns hypothesis[22–24] (every real number that can be real-time generated is either rational or transcendental) still remains unresolved. Next, the transcendental numbers mentioned in the just referenced result are Liouville numbers[22] and it is known that these numbers possess very long series with the same binary symbol in their binary expansion. We do not know whether the arithmetical space $\mathcal{B}$ contains real time computable[22] transcendental numbers. If not, then it follows from Hartmanis-Stearns hypothesis and the above-introduced definition of theoretical spike train, that spontaneous neuron spike trains are not computable in real time. Thus, we have arrived at the next conventional statement: if the just mentioned hypotheses are true, it is impossible to generate (in real time) very long series of spontaneous neuron spike trains.

## 5. CONCLUSIONS

Based on computational analysis of experimental data on neuron activity, the paper introduces the notion of conjugate spike train in terms of which two determining properties of actual neuron spike trains are postulated. In this way, we give the definitions of a mathematical single neuron and introduce the brain activity attractor $\hat{\mathcal{B}}$, asymptotically close to where the brain activity is found and functions in the case when the stationary conditions hold during long enough time periods. With this end in view, the existence of some attractor $\hat{\mathcal{B}}^* \subset \mathcal{B}^{\infty}$, which is contained in multidimensional Cantor set with zero Lebesgue measure and is conjugate to $\hat{\mathcal{B}}$ is established. Here, $\mathcal{B}$ is a collection of all numbers having bounded series with the same binary symbol in their binary expansion. The existence of attractor $\hat{\mathcal{B}}$ indicates an interesting direction of brain research, where the methods and results of fractals theory and multifractal analysis can be applied. Apparently, the first problem arising here is the determination of fractal (or multifractal) dimensions of the brain activity attractor for various biological species.

The work also establishes the property of self-affinity of neuron spike train records as well as their power law behavior. The possibility of (real-time) computer generation of neuron spike trains in connection with some unresolved problems of computations theory is discussed.

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